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Three-dimensional flow near surface distortions for the compensation regime

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Flow past an isolated, small, three-dimensional roughness located on a plate is investigated theoretically and numerically. Depending on roughness scales, there are different local disturbed flow regimes. The so-called 'compensation regime' is characterized by conservation of boundary-layer thickness for which the flow near the roughness does not interact with the external inviscid flow. A spectral numerical method is used to calculate the corresponding three-dimensional boundary problem.

Keywords: fluid dynamics; high Reynolds number; surface distortions; three-dimensional boundary-layer flows

1. Introduction

Phenomena arising in the flow near surface distortions play a significant role in geophysical flows, in external and internal aerodynamics, etc. Modern trends in boundary-layer flow control give the basis of a more detailed analysis. The laminar-turbulent transition, skin friction and heat transfer distributions depend strongly on surface irregularities. By means of controlled unsteady motion of surface irregularities the transition may be delayed or boundary-layer separation may be suppressed.

Different local flow regimes are described by Smith (1973, 1976), Bogolepov & Neiland (1971), Bogolepov (1986), Duck & Burggraf (1986), Bogolepov & Lipatov (1985) and Sykes (1980). We do not aim to discuss all aspects of locally disturbed flows. A more comprehensive list of publications is presented, for example, by Smith & Walton (1998) and by Roget *et al.* (1998). Investigations of two-dimensional flows near surface distortions provided by Bogolepov & Neiland (1971), analysis of pipeflows done by Smith (1976) and stratified flows studies made by Sykes (1980) allow us to find the so-called compensation regime. This regime is characterized by conservation of boundary-layer thickness and corresponds to a variety of disturbed flows.

At the same time the numerical solutions for nonlinear problems were obtained by Sykes (1980) only for flows periodical in the transverse direction. Not all the problems were solved from a theoretical point of view. This paper is devoted to the analysis of mathematical and physical aspects of locally disturbed flows. The aim is to describe the main processes responsible for disturbance propagation and to formulate a well-posed mathematical problem describing the compensation regime.

This paper is also devoted to the numerical investigation of the flow near three-dimensional surface distortion for the compensation regime.

2. Problem formulation

The flow near the flat plate is considered. It is supposed that the small surface distortion is located on the bottom of a laminar two-dimensional boundary layer at a distance l from the flat plate leading edge. It is supposed that the external flow is uniform subsonic or supersonic viscous flow $(M_\infty^2 - 1) \sim 1$ for large but subcritical Reynolds numbers $Re_\infty = \rho_\infty u_\infty l/\mu_\infty = \varepsilon^{-2}$; ρ, u and μ are the density, the velocity and the dynamical viscosity coefficient, respectively; the index ∞ corresponds to the values in the external flow. Cartesian coordinates are introduced, where X, Y and Z are the streamwise, normal and spanwise axes, respectively. We will use the following definitions below: lt/u_∞, lx, ly, lz, $u_\infty u$, $u_\infty v$, $u_\infty w$, $\rho_\infty u_\infty^2$, $\rho_\infty \rho$, $\mu_\infty \mu$ for time, coordinates, velocity components, pressure, density and dynamical viscosity coefficient, respectively. It is supposed that the distortion has the scale-thickness a, length b and width c.

Let us consider the flow near the local distortion having comparable length and width $b \sim c$. It is evident that the limiting problems for asymptotically different values characterizing the width and the length may be obtained as a result of the corresponding limiting procedure from the general mathematical problem investigated.

It is also supposed that the characteristic length and the width of the distortion are much larger than the undisturbed boundary-layer thickness in the vicinity of the distortion. In accordance with the matched asymptotic expansions method, we need to introduce the region 1 having asymptotically equal scales in all dimensions. It may be shown for small thickness values that the disturbed flow is characterized by equal values of disturbances for velocity components, pressure and density, which are determined by the vertical velocity value on the external edge of the boundary layer. This required value may be estimated as the ratio of characteristic height to characteristic length:

$$\Delta p \sim v \sim a/b. \tag{2.1}$$

This estimate allows us to determine the boundary-layer thickness change induced by the pressure disturbance. The longitudinal velocity near the distortion in the undisturbed boundary layer at a height comparable with that of the distortion is determined by the following expression:

$$u \sim a/\varepsilon. \tag{2.2}$$

If the distortion induces nonlinear longitudinal velocity changes, then the following relations are valid,

$$u \sim \Delta u \sim \Delta p^{1/2}, \tag{2.3}$$

$$y \sim \Delta y \sim \varepsilon \Delta p^{1/2}, \tag{2.4}$$

where the last relation determines the nonlinear region thickness. The vertical velocity on the external boundary-layer edge is induced both by the distortion height and the change in boundary-layer thickness. It may then be shown that the total change in boundary-layer thickness is determined to leading order by the region located near the wall where streamlines have nonlinear changes in longitudinal velocity. The pressure disturbance estimate may be written in the form:

$$\Delta p \sim (a/b) + \varepsilon \Delta p^{1/2}/b. \tag{2.5}$$

As was shown by Bogolepov & Neiland (1971), this relation is valid only if

$$b/a \sim b^2/\varepsilon^2. \tag{2.6}$$

The other limit, corresponding to

$$b^2/\varepsilon^2 \ll b/a, \tag{2.7}$$

leads to the discrepancy that the change in boundary-layer thickness induces a much larger pressure disturbance than the original pressure disturbance leading to the thickness change. The problem may be resolved if disturbances in the external flow are absent (to leading order). Therefore, instead of relation (2.5), the following compensation relation (zero change in total boundary-layer thickness) will be used:

$$a/b \sim \varepsilon \Delta p^{1/2}/b, \tag{2.8}$$

$$\Delta p \sim a^2/\varepsilon^2. \tag{2.9}$$

It is necessary to also take into account equal orders of viscosity and inertia forces in nonlinearly disturbed region. This condition follows from the longitudinal momentum equation analysis,

$$a \sim \varepsilon b^{1/3}. \tag{2.10}$$

All estimates obtained earlier lead to the inequality,

$$b \ll \varepsilon^{3/4}, \tag{2.11}$$

determining the length of distortion when compensation regime take place. The equality in (2.11) corresponds to the free interaction regime.

The conservation of boundary-layer thickness may be written in the form,

$$u = Ay\varepsilon^{-1} + o(1), \tag{2.12}$$

where A is the non-dimensional skin friction on the surface in the undisturbed boundary layer upstream from the distortion. This condition follows from the matching procedure applied to solutions in the region 2 (the main part of the boundary-layer flow) and in the region 3 (nonlinearly disturbed region near the wall).

3. Boundary value problem

In region 3, which has length-scales $x \sim b$, $y \sim \varepsilon b^{1/3}$, $z \sim c$, the following asymptotic expansions are introduced:

$$x = bx_3, \quad y = \varepsilon b^{1/3}\mu_w^{1/3}\rho_w^{-1/3}A^{-1/3}y_3, \quad z = cz_3, \tag{3.1 a}$$

$$u = A^{1/3}\mu_w^{1/3}\rho_w^{-1/3}b^{1/3}u_3 + \cdots, \tag{3.1 b}$$

$$v = A^{1/3}\mu_w^{2/3}\rho_w^{-2/3}\varepsilon b^{-1/3}v_3 + \cdots, \tag{3.1 c}$$

$$w = A^{2/3}\mu_w^{1/3}\rho_w^{-1/3}b^{-2/3}cw_3 + \cdots, \tag{3.1 d}$$

$$p = 1/\gamma M_\infty^2 + A^{4/3}\mu_w^{2/3}\rho_w^{1/3}b^{2/3}p_3 + \cdots, \quad \rho = \rho_w + \cdots, \quad \mu = \mu_w + \cdots. \tag{3.1 e}$$

The limiting relations, $a \sim \varepsilon b^{1/3}$, $\varepsilon^{3/2} < b < \varepsilon^{3/4}$, $c \sim b$, then correspond to the following equations, deduced from the Navier–Stokes equations for leading terms in (3.1):

$$\frac{\partial u_3}{\partial x_3} + \frac{\partial v_3}{\partial y_3} + \frac{\partial w_3}{\partial z_3} = 0, \tag{3.2 a}$$

$$u_3\frac{\partial u_3}{\partial x_3} + v_3\frac{\partial u_3}{\partial y_3} + w_3\frac{\partial u_3}{\partial z_3} + \frac{\partial p_3}{\partial x_3} = \frac{\partial^2 u_3}{\partial y_3^2}, \tag{3.2 b}$$

$$\frac{\partial p_3}{\partial y_3} = 0, \tag{3.2 c}$$

$$u_3\frac{\partial w_3}{\partial x_3} + v_3\frac{\partial w_3}{\partial y_3} + w_3\frac{\partial w_3}{\partial z_3} + D\frac{\partial p_3}{\partial z_3} = \frac{\partial^2 w_3}{\partial y_3^2}, \tag{3.2 d}$$

where the similarity parameter $D = b^2/c^2$ is determined by the ratio of length to width. Boundary conditions include the usual conditions for the three-dimensional boundary layer along with total zero thickness change condition,

$$u_3 = v_3 = w_3 = 0 \quad \text{at } y_3 = hf(x_3, z_3), \tag{3.3 a}$$

$$u_3 \to y_3, \quad v_3, \quad w_3, \quad p_3 \to 0 \quad \text{as } x_3 \to -\infty, \quad z_3 \to \pm\infty, \tag{3.3 b}$$

$$u_3 \to y_3, \quad w_3 \to 0 \quad \text{as } y_3 \to \infty, \tag{3.3 c}$$

where $h = A^{1/3}\mu_{\mathrm{w}}^{-1/3}\rho_{\mathrm{w}}^{1/3}ab^{-1/3}\varepsilon^{-1}$ (the subscripts are suppressed below).

The boundary-value problem (3.2)–(3.3) contains two similarity parameters, D and h. When the first parameter tends to zero, problem (3.2)–(3.3) is reduced to the boundary-value problem describing two-dimensional flow.

The second similarity parameter is proportional to the inertia and viscosity forces ratio in region 3. For large h values, the disturbed flow near the distortion is inviscid at leading order; correspondingly, small h values are connected with the influence of the viscosity force and with linearly disturbed flow near the distortion.

4. Linear solution

Small values of h correspond to a linear solution which may be sought in the form,

$$u = y + hU + \cdots, \qquad v = hV + \ldots, \qquad p = hP + \cdots, \qquad w = hW + \cdots. \tag{4.1}$$

Corresponding linearized equations for $D = 1$ have the form,

$$y\frac{\partial U}{\partial x} + V + \frac{\partial P}{\partial x} = \frac{\partial^2 U}{\partial y^2}, \tag{4.2 a}$$

$$y\frac{\partial W}{\partial x} + \frac{\partial P}{\partial z} = \frac{\partial^2 W}{\partial y^2}, \tag{4.2 b}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{4.2 c}$$

$$\frac{\partial P}{\partial y} = 0. \tag{4.2 d}$$

Equations (4.2) may be transformed to the following equations:

$$y\frac{\partial S}{\partial x} = \frac{\partial^2 S}{\partial y^2}, \tag{4.3 a}$$

$$-\frac{\partial S(x,0,z)}{\partial y} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2}, \tag{4.3 b}$$

$$\int_0^\infty S \,\mathrm{d}y = -\frac{\partial f}{\partial x}, \tag{4.3 c}$$

where $S = \partial^2 V/\partial y^2$. It may be deduced from the analysis of (4.3) that a non-trivial solution for the function S exists if $f \neq 0$ or if there is a non-zero convective derivative of the function S (the wake downstream from the distortion). Then for an arbitrary surface point, excluding the distortion and its wake, the solution of this equation has the form,

$$S(x,y,z) = 0, \qquad V(x,y,z) = 0. \tag{4.4}$$

For the velocity components the following problem may be deduced,

$$y\frac{\partial U}{\partial x} + \frac{\partial P}{\partial x} = \frac{\partial^2 U}{\partial y^2}, \tag{4.5 a}$$

$$y\frac{\partial W}{\partial x} + \frac{\partial P}{\partial z} = \frac{\partial^2 W}{\partial y^2}, \tag{4.5 b}$$

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \tag{4.5 c}$$

$$\frac{\partial P}{\partial y} = 0, \tag{4.5 d}$$

which describes the quasi-two-dimensional flow outside the distortion and its wake. Equations (4.5) also follow from the analysis made by Smith (1976) and by Bogolepov & Lipatov (1985).

5. Analysis of the nonlinearly disturbed flow

Wang (1971) described how to analyse boundary-layer equations. Following Wang, we will introduce a subcharacteristic surface $\Omega(x,y,z)$ and will transform the independent variables,

$$x, y, z \to \Omega(x,y,z), y, z. \tag{5.1}$$

As a result of this analysis we may obtain the following equation, which determines the subcharacteristics,

$$\left(\frac{\partial \Omega}{\partial y}\right)^3 \left[\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial z}\right)^2\right] \left(u\frac{\partial \Omega}{\partial x} + v\frac{\partial \Omega}{\partial y} + w\frac{\partial \Omega}{\partial z}\right) = 0. \tag{5.2}$$

The first multiplier corresponds to the characteristics of the original equation. The second multiplier corresponds to the elliptical type of equation for the pressure, and the third multiplier states that the vorticity (velocity) field is controlled by streamlines. This form determines the conditions needed to formulate a well-posed boundary-value problem for the three-dimensional boundary layer with compensation interaction regime.

Thus there are three main mechanisms of disturbance propagation in the flow corresponding to that near a distortion. The first mechanism is diffusion, in which disturbances propagate in a normal direction with infinite speed. The second mechanism is determined by pressure, in which disturbances also propagate with infinite speed. The third mechanism is controlled by convection. All these mechanisms are not only connected with physical processes but are also connected with the mathematical formulation of the problem. The mathematical problem will be well-posed if all mechanisms of disturbance propagation are precisely taken into account, for example, in the numerical method. Otherwise, some form of instability may be encountered in the numerical procedure. Analysing the system of equations, we may conclude that, as in the linear case, there is a region of zero vertical velocity and a region where this velocity is non zero.

Let us suppose that, as in the linear case, there is a region of flow with zero vertical velocity near a finite distortion. Then the disturbed flow is described by the following equations:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \tag{5.3 a}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2}, \tag{5.3 b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{5.3 c}$$

Equations (5.3) may be transformed as follows,

$$u\frac{\partial \omega_y}{\partial x} + w\frac{\partial \omega_y}{\partial z} = \frac{\partial^2 \omega_y}{\partial y^2}, \tag{5.4}$$

where

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

Equation (5.4) is accompanied by uniform boundary conditions,

$$\omega_y = 0 \quad \text{at } y = 0, \tag{5.5}$$

$$\omega_y \to 0 \quad \text{as } y \to \infty. \tag{5.6}$$

Then a zero solution for vorticity component ω_y exists for the region of flow analysed if

$$\omega_y = 0. \tag{5.7}$$

Applying the divergence operator to the equations (5.3) allows us to obtain the following equation:

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} = -\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial z^2}. \tag{5.8}$$

Equations (5.3) and (5.4) allow to transform (5.7) as follows:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = -\tfrac{1}{2}\Delta p. \tag{5.9}$$

Only the zero solution for the velocity component then exists. But this conclusion contradicts equation (5.3 *b*). Therefore, in the nonlinear case, the suggestion that the normal velocity is zero in part of the disturbed flow is not true. This suggestion is fulfilled only for the linear case.

6. Numerical solution

The spectral method originally suggested by Duck & Burggraf (1986) was used to solve the problem (3.2)–(3.3). Prandtl's transposition,

$$\tilde{x} = x, \quad \tilde{y} = y - f(x,z), \quad \tilde{z} = z, \quad \tilde{u} = u - \tilde{y}, \tag{6.1 a}$$

$$\tilde{v} = v - u\frac{\partial f}{\partial x} - w\frac{\partial f}{\partial z}, \quad \tilde{w} = w, \quad \tilde{p} = p, \tag{6.1 b}$$

leads to the problem including unchanged equations and the following boundary conditions:

$$u = v = w = 0 \quad \text{at } y = 0, \tag{6.2 a}$$

$$u \to 0, \quad v, w, p \to 0 \quad \text{as } x \to -\infty \text{ and as } z \to \pm\infty, \tag{6.2 b}$$

$$u \to f(x,z), \quad w \to 0 \quad \text{as } y \to \infty. \tag{6.2 c}$$

Fourier transforms in the x- and z-directions give the following expression for the longitudinal velocity:

$$u^{**}(k,l,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x,y,z)\exp(-\mathrm{i}kx - \mathrm{i}lz)\,\mathrm{d}x\,\mathrm{d}z. \tag{6.3}$$

The system to be solved is written then as follows:

$$\mathrm{i}ku^{**} + v^{**\prime} + \mathrm{i}lw^{**} = 0, \tag{6.4 a}$$

$$u^{**\prime\prime} - \mathrm{i}kyu^{**} - v^{**} - \mathrm{i}kP = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)^{**} = R_1^{**}, \tag{6.4 b}$$

$$w^{**\prime\prime} - \mathrm{i}kyw^{**} - \mathrm{i}lP^{**} = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)^{**} = R_2^{**}. \tag{6.4 c}$$

The boundary conditions include

$$u^{**} = v^{**} = w^{**} = 0 \quad \text{at } y = 0, \tag{6.5 a}$$

$$u^{**} \to F^{**}, \quad w^{**} \to 0 \quad \text{as } y \to \infty. \tag{6.5 b}$$

The following procedure is mainly the same as that described by Duck & Burggraf (1986). The distortion analysed has the geometry $f(x,z) = h\exp(-(x^2+z^2))$. The numerical grid includes $NX \times NY \times NZ = 64 \times 26 \times 32$ nodes, with the steps for corresponding coordinates $\Delta x = \Delta z = 0.3$, $\Delta y = 0.4$, $D = 1$. To check out the accuracy of results, the grid $NX \times NY \times NZ = 128 \times 26 \times 64$ was used along with non-uniform spacing in the Y-direction. The results obtained on both grids were almost the same. To have a uniformly valid solution of linearized equations, integration in spectral space was fulfilled by analytical presentation of functions near the singular point $k, l = 0$.

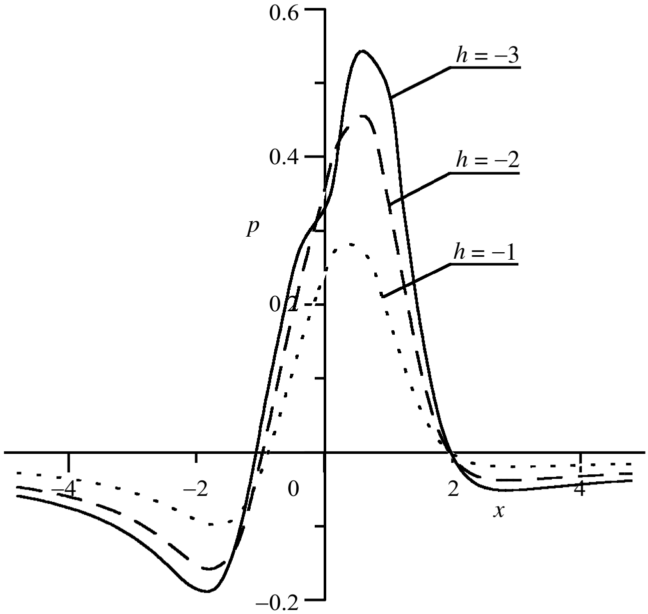

Figure 1. Pressure distributions in the symmetry plane.

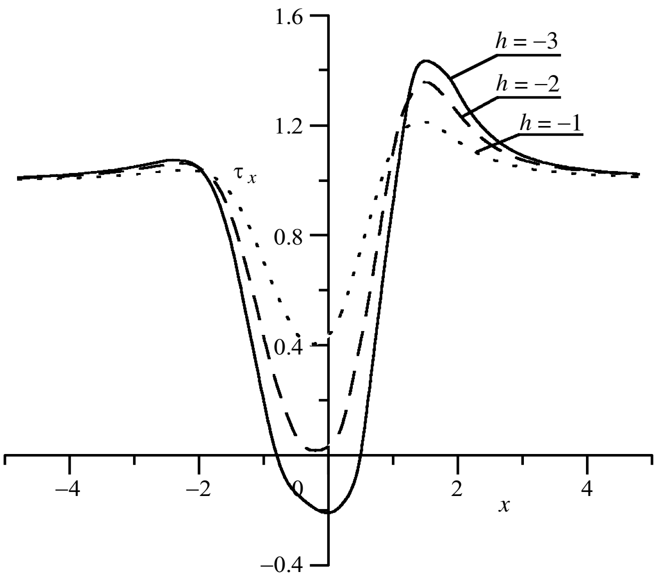

Figure 2. Longitudinal skin friction distributions in the symmetry plane.

In figure 1 the pressure disturbance distributions are presented in the symmetry plane for different h values. The negative h values correspond to the hollow on the surface. It may be seen that the distortion depth increase leads to the maximum pressure disturbance rise. The distributions presented are characterized by two local minima located downstream and upstream from the coordinate origin.

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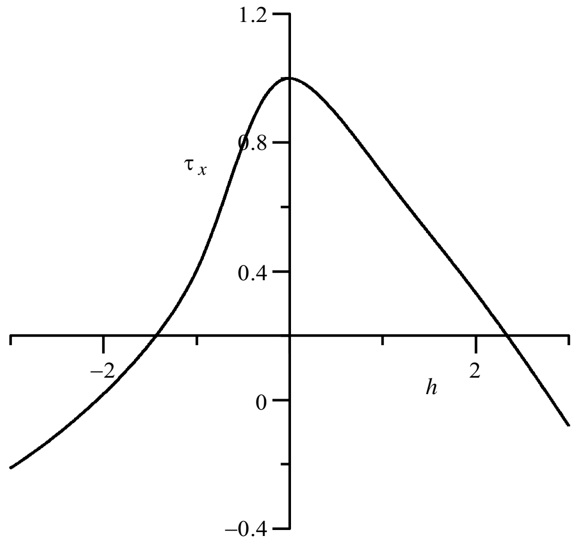

Figure 3. Minimal skin friction distribution.

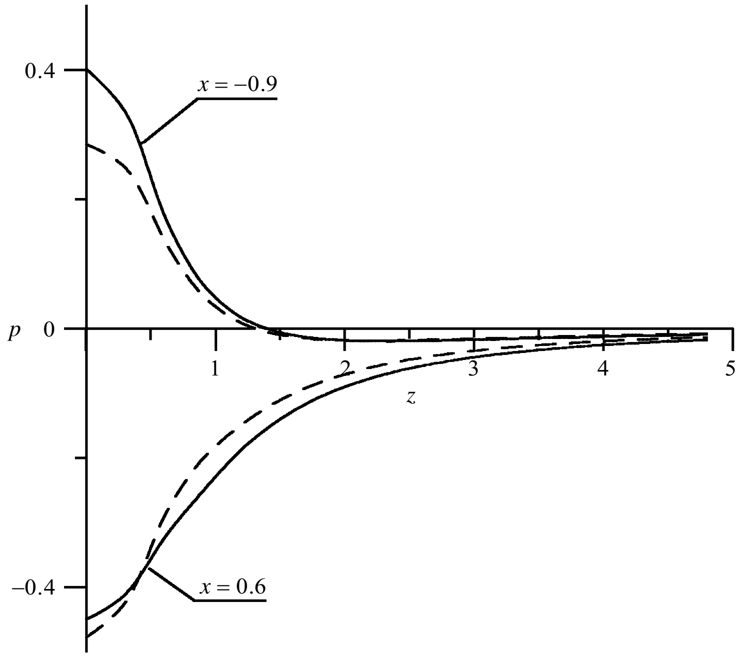

Figure 4. Pressure distributions in the transversal direction.

It is worth mentioning that there is tendency in the pressure distribution to form a plateau region.

Results of the longitudinal skin friction calculations are presented on figure 2. It may be concluded that the distortion depth increase leads to the minimum skin friction diminishing. A limiting depth value exists for which the longitudinal skin friction equals zero at one surface point.

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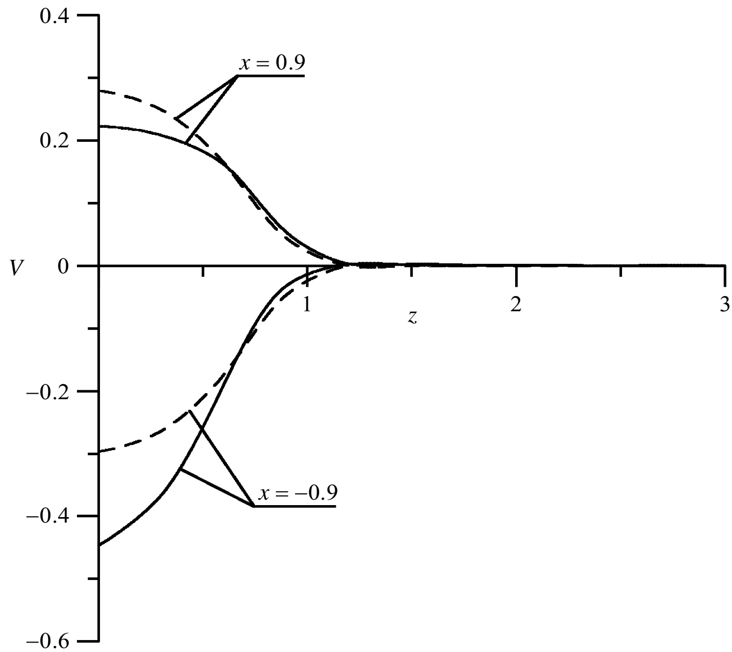

Figure 5. Vertical velocity distributions.

The results presented in figure 3 illustrate the minimum longitudinal skin friction dependence on the parameter h. It may be supposed that, for large depth values, a limiting regime exists for which minimal longitudinal skin friction tends to a finite value. The distortion height increase leads to a different tendency. At the same time, further investigation is needed to arrive at our final conclusions.

The disturbed flow near finite distortions of the form

$$f(x,z)=0, \quad \text{if } R \geqslant 1 \quad \text{and} \quad f(x,z)=h\cos^2(\pi R/2) \quad \text{for } R<1,$$

where $R = x^2 + z^2$, was also investigated.

In figure 4 the dependence of pressure disturbance is presented as a function of z for $h = 1.5$ in the linear and nonlinear case (the solid line corresponds to the nonlinear case) for different longitudinal coordinate values. It may be seen that, in accordance with the aforementioned results, the distortion influence is revealed in the full flow field, due to the ellipticity of the equation for the pressure disturbance.

In figure 5 the vertical velocity distribution is presented in the plane parallel to the flat plate. It may be seen in fact that outside the influence zone the vertical velocity almost equals zero. This result does not contradict to the conclusion made in § 5. When the distortion height is not so large, nonlinear effects outside the distortion are small.

It is worth mentioning that the mathematical model investigated may be used in a wide class of problems. The results obtained, for example, may be used to estimate the effectiveness of flow control in three-dimensional laminar boundary layers.

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